Chapter 3

Riesz transforms on R^d

3.1 Fourier Integrals.

We now look at Fourier Transforms on R^d. If $f(x)$ is a function in $L_1(R^d)$ its Fourier transform $\widehat{f}(y)$ is defined by

$$\widehat{f}(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i<x,y>} f(x)dx \tag{3.1}$$

We denote by $\mathcal{S}$ the class of all functions f on R^d that are infinitely differentiable such that the function and its derivitives of all orders decay faster than any power, i.e. for every $n_1, n_2, \ldots, n_d \geq 0$ and $k \geq 0$ there are constants $C_{n_1,n_2,\ldots,n_d,k}$ such that

$$|[(\frac{d}{dx_1})^{n_1}(\frac{d}{dx_1})^{n_2} \ldots (\frac{d}{dx_d})^{n_d} f](x)| \leq C_{n_1,n_2,\cdots,n_d,k}(1 + \|x\|)^{-k}$$

It is easy to show (left as an exercise) by repeated integration by parts that if $f \in \mathcal{S}$ so does $\widehat{f}$.

Theorem 3.1. *The Fourier transform has the inverse*

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \widehat{f}(y)dy \tag{3.2}$$

proving that the Fourier transform is a one to one mapping of $\mathcal{S}$ onto itself.

In addition the Fourier transform extends as a unitary map from $L_2(R^d)$ onto $L_2(R^d)$.

Proof. Clearly

$$g(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \widehat{f}(y) dy$$

is well defined as a function in $\mathcal{S}$. We only have to identify it. We compute g as

$$\begin{aligned} g(x) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \widehat{f}(y) dy \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \widehat{f}(y) e^{-\epsilon \frac{\|y\|^2}{2}} dy \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} \left[\left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i<z,y>} f(z) dz \right] e^{-i<x,y>} e^{-\epsilon \frac{\|y\|^2}{2}} dy \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{2\pi}\right)^d \int_{R^d} \int_{R^d} e^{i<z-x,y>} f(z) e^{-\epsilon \frac{\|y\|^2}{2}} dy dz \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{2\pi}\right)^d \int_{R^d} f(z) \left[\int_{R^d} e^{i<z-x,y>} e^{-\epsilon \frac{\|y\|^2}{2}} dy \right] dz \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi\epsilon}}\right)^d \int_{R^d} f(z) e^{-\frac{\|z-x\|^2}{2\epsilon}} dz \\ &= f(x) \end{aligned}$$

Here we have used the identity

$$\frac{1}{\sqrt{2\pi}} \int_R e^{i\,xy} e^{-\frac{x^2}{2}} dx = e^{-\frac{y^2}{2}}$$

We now turn to the computation of L_2 norm of $\widehat{f}$. We calculate it as

$$\begin{aligned}
\|\widehat{f}\|_2^2 &= \lim_{\epsilon\to 0}\int_{R_d} |\widehat{f}(y)|^2 e^{-\frac{\epsilon\|y\|^2}{2}} dy \\
&= \lim_{\epsilon\to 0}\int_{R_d}\int_{R_d}\int_{R_d} f(x)\bar{f}(z) e^{i<x-z,y>} e^{-\frac{\epsilon\|y\|^2}{2}} dydxdz \\
&= \lim_{\epsilon\to 0}\left(\frac{1}{\sqrt{2\pi\epsilon}}\right)^d \int_{R_d}\int_{R_d} f(x)\bar{f}(z) e^{-\frac{\|x-z\|^2}{2\epsilon}} dxdz \\
&= \lim_{\epsilon\to 0}\int_{R^d} f(x)[K_\epsilon \bar{f}](x)dx \\
&= \int_{R^d} |f(x)|^2 dx
\end{aligned}$$

Since the $f \to \widehat{f}$ preserves the L_2 norm and is onto from $\mathcal{S} \to \mathcal{S}$, it extends to the completion L_2 as a unitary map. □

We see that the Fourier transform is a bounded linear map from L_1 to L_∞ as well as L_2 to L_2 with corresponding bounds $C = (\frac{1}{\sqrt{2\pi}})^d$ and 1. By the Riesz-Thorin interpolation theorem (see the exercise in Chapter 2) the Fourier transform is bounded from L_p into $L_{\frac{p}{p-1}}$ for $1 \le p \le 2$. If $\frac{1}{p} = 1.t + \frac{1}{2}(1-t)$ then $\frac{1}{2}(1-t) = 1 - \frac{1}{p} = \frac{p-1}{p}$. See exercise to show that, for $f \in L_p$ with $p > 2$, the Fourier Transform need not exist.

For convolution operators of the form

$$(Tf)(x) = (k * f)(x) = \int_{R^d} k(x-y)f(y)dy \tag{3.3}$$

we want to estimate $\|T\|_p$, the operator norm from L_p to L_p for $1 \le p \le \infty$. As before for $p = 1, \infty$,

$$\|T\|_p = \int_{R^d} |k(y)|dy.$$

Let us suppose that for some constant C,

1. The Fourier transform $\widehat{k}(y)$ of $k(\cdot)$ satisfies

$$\sup_{y\in R^d} |\widehat{k}(y)| \le C < \infty \tag{3.4}$$

2. In addition,

$$\sup_{x \in R^d} \int_{\{y : \|x-y\| \geq C\|x\|\}} |k(y-x) - k(y)| dy \leq C < \infty \tag{3.5}$$

We will estimate $\|T\|_p$ in terms of C. The main step is to establish a weak type $(1,1)$ inequality. Then we will use the interpolation theorems to get boundedness in the range $1 < p \leq 2$ and duality to reach the interval $2 \leq p < \infty$.

Theorem 3.2. *The function $g(x) = (Tf)(x) = (k * f)(x)$ satisfies a weak type $(1,1)$ inequality*

$$\mu\{x : |g(x)| \geq \ell\} \leq C_0 \frac{\|f\|_1}{\ell} \tag{3.6}$$

with a constant C_0 that depends only on C.

We first prove a decomposition lemma that we will need for the proof of the theorem.

Lemma 3.3. *Given any open set $G \in R^d$ of finite Lebesgue measure we can find a countable set of balls $\{S(x_j, r_j)\}$ with the following properties. The balls are all disjoint. $G = \cup_j S(x_j, 3r_j)$ is the countable union of balls with the same centers but three times the radius. More over there is a number $k_1(d)$ that depends only on the dimension such that each point of G is covered at most $(96)^d$ times by the covering $G = \cup_j S(x_j, 3r_j)$. Finally each of the balls $S(x_j, 5r_j)$ has a nonempty intersection with G^c.*

Basically, the lemma says that it is possible to write G as a nearly disjoint countable union of balls each having a radius that is comparable to the distance of its center to the boundary.

Proof. Suppose G is an open set in the plane of finite volume.

Let $d(x) = d(x, G^c)$ be the distance from x to G^c or the boundary of G. Let $d_0 = \sup_{x \in G} d(x)$. Since the volume of G is finite, G cannot contain any large balls and consequently d_0 cannot be infinite.

We consider balls $S(x, r(x))$ around x of radius $r(x) = \frac{d(x)}{4}$. They are contained in G and provide a covering of G as x varies over G. All these balls have the property that $S(x, 3r(x)) \subset G$ and $S(x, 5r(x))$ intersects G^c.

We proceed to select a countable collection $\{S(x_i, r(x_i))\}$ from $\{S(x, r(x)\}$ that are disjoint while $\cup_i S(x_i, 3r(x_i)) = G$.

We choose x_1 such that $d(x_1) > \frac{d_0}{2}$. Having chosen $x_1, \ldots, x_k$ the choice of x_{k+1} is made as follows. We consider the balls $S(x_i, r(x_i))$ for $i = 1, 2, \ldots, k$. Look at the set $G_k = \{x : S(x, r(x)) \cap S(x_i, r(x_i)) = \emptyset \text{ for } 1 \le i \le k\}$ and define $d_k = \sup_{x \in G_k} d(x)$. We pick $x_{k+1} \in G_k$ such that $d(x_{k+1}) > \frac{d_k}{2}$. We proceed in this fashion to get a countable collection of balls $\{S(x_i, r(x_i))\}$.

By construction, they are disjoint balls contained in the set G of finite volume and therefore $r(x_i) \to 0$ as $i \to \infty$. Since, $d_i \le 2d(x_{i+1}) \le 8r(x_{i+1})$ it follows that $d_i \to 0$ as $i \to \infty$. Every $S(x_i, 5r(x_i))$ intersects G^c.

We now examine how much of G the balls $\{B(x_i, r(x_i))\}$ cover. First we note that

$$G_0 \supset G_1 \supset \cdots \supset G_k \supset G_{k+1} \supset \cdots$$

We claim that $\cap_k G_k = \emptyset$. If not, let $x \in G_k$ for every k. Then $d_k = \sup_{y \in G_k} d(y) \ge d(x) > 0$ for every k. This contradicts the convergence of d_k to 0.

Since $x \in G_0 = G$, we can find $k \ge 1$ such that $x \in G_{k-1}$ but $x \notin G_k$. Then $S(x, r(x))$ must intersect $S(x_k, r(x_k))$ giving us the inequality

$$|x - x_k| < r(x) + r(x_k) \le \frac{d(x)}{4} + r(x_k) \le \frac{d_{k-1}}{4} + r(x_k) \le \frac{d(x_k)}{2} + r(x_k) = 3r(x_k)$$

Clearly $S(x_k, 3r(x_k))$ will contain x. Since $3r(x) < d(x)$ the enlarged ball is still within G. This means $G = \cup_k S(x_k, 3r(x_k))$.

Now we will bound the number of times a point x can be covered by $\{S(x_k, 3r(x_k)\}$. Let for some k, $|x - x_k| < 3r(x_k)$. Then by the triangle inequality

$$|d(x) - d(x_k)| \le 3r(x_k)$$

or equivalently (recall $r(x) = \frac{d(x)}{4}$)

$$|r(x) - r(x_k)| \le \frac{3}{4} r(x_k)$$

This implies that for the ratio $|\frac{r(x)}{r(x_k)} - 1| \le \frac{3}{4}$ we have $\frac{1}{4} \le \frac{r(x)}{r(x_k)} \le \frac{7}{4}$ In particular any ball $S(x_k, 3r(x_k))$ that covers x, must have its center with in a distance of $3r(x_k) \le 12r(x)$ of x and the corresponding $r(x_k)$ must be in the range $\frac{4}{7}r(x) \le r(x_k) \le 4r(x)$. The balls $S(x_k, r(x_k))$ are then contained in $S(x, 24r(x))$ are disjoint and have a radius of at least $\frac{4}{7}r(x)$. There can be at most $k_1(d) = (42)^d$ of them by considering the total volume. We can choose our norm in R^d to be $\max_i |x_i|$ and force the spheres to be cubes.

□

Proof of theorem. The proof is similar to the one-dimensional case with some modifications.

1. We let G_ℓ be the open set where the maximal function $M_f(x)$ satisfies $|M_f(x)| > \ell$. From the maximal inequality
$$\mu[G_\ell] \le C(d)\frac{\|f\|_1}{\ell} \tag{3.7}$$

2. We write $G_\ell = \cup_i B_i = \cup_i S(x_i, 3r(x_i))$, a countable union of spheres according to the lemma.

3. If we let
$$\phi(x) = \sum_i \mathbf{1}_{B_i}(x)$$
then $1 \le \phi(x) \le k_1(d)$ on G_ℓ.

4. Let us define a weighted average m_i of $f(y)$ on B_i by
$$\int_{B_i} [f(y) - m_i]\frac{dy}{\phi(y)} = 0 \tag{3.8}$$
and write

$$\begin{aligned} f(x) &= f(x)\mathbf{1}_{G_\ell^c}(x) + \frac{1}{\phi(x)}\sum_i f(x)\mathbf{1}_{B_i}(x) \\ &= \left[f(x)\mathbf{1}_{G_\ell^c}(x) + \frac{1}{\phi(x)}\sum_i m_i\mathbf{1}_{B_i}(x)\right] \qquad (3.9) \\ &\quad + \frac{1}{\phi(x)}\sum_i [f(x) - m_i]\mathbf{1}_{B_i}(x) \\ &= h_0(x) + \sum_i h_i(x) \qquad (3.10) \end{aligned}$$

5. For any sphere B_i with center x_i and radius $3r(x_i)$ there is a sphere with radius $5(r(x_i))$ with the same center that contains a point $x'_i \in G^c_\ell$ with $|M_f(x'_i)| \leq \ell$. The sphere $\widehat{B}_i = S(x'_i, 8r(x_i))$ contains B_i. Since $1 \leq \phi(y) \leq k_1(d)$ on G_ℓ

$$\begin{aligned}
|m_i| &\leq \left[\int_{B_i} \frac{|f(y)|}{\phi(y)} dy\right]\left[\int_{B_i} \frac{1}{\phi(y)} dy\right]^{-1} \\
&\leq k_1(d) \frac{1}{\mu(B_i)} \int_{B_i} |f(y)| dy \\
&= k_1(d) \big(\frac{8}{3}\big)^d \frac{1}{\mu(\widehat{B}_i)} \int_{B_i} |f(y)| dy \leq k_2(d) \\
&\leq k_2(d) \frac{1}{\mu(\widehat{B}_i)} \int_{\widehat{B}_i} |f(y)| dy \\
&\leq k_2(d) M_f(x'_i) \\
&\leq k_2(d)\ell
\end{aligned}$$

It follows that on G_ℓ

$$\frac{1}{\phi(x)} \sum_i m_i \mathbf{1}_{B_i}(x) \leq k_2(d)\ell$$

Moreover on G^c_ℓ, $|f(x)| \leq M_f(x) \leq \ell$. Since $k_2(d) \geq 1$

$$\|h_0\|_\infty \leq \max\{1, k_2(d)\}\ell = k_2(d)\ell \tag{3.11}$$

On the other hand $\phi(x) \geq 1$ on G_ℓ and

$$\begin{aligned}
\|h_0\|_1 &\leq \|f\|_1 + k_2(d)\ell \sum_i \mu[B_i] \\
&\leq \|f\|_1 + k_2(d)\ell\mu[G_\ell] \\
&\leq (1 + k_2(d))\|f\|_1
\end{aligned} \tag{3.12}$$

and therefore

$$\|h_0\|_2^2 \leq \|h_0\|_1 \|h_0\|_\infty \leq k_3(d)\ell\|f\|_1 \tag{3.13}$$

From the boundedness of T from L_2 to L_2 this gives

$$\mu\{x : |(Th_0)(x)| \geq \ell\} \leq \frac{\|Th_0\|_2^2}{\ell^2} \leq C^2 k_3(d) \frac{\|f\|_1}{\ell} \tag{3.14}$$

where C is the bound on $|\widehat{k}|$ from (3.4)

6. We now turn our attention to the functions $\{h_j\}$

$$w(x) = \sum_i (Th_i)(x) = \sum_i \int_{B_i} [f(y) - m_j] k(x-y) \frac{dy}{\phi(y)}$$
$$= \sum_i \int_{B_i} [f(y) - m_i][k(x-y) - k(x-x_i)] \frac{dy}{\phi(y)}$$

$$|w(x)| \leq \sum_i \int_{B_i} |f(y) - m_i||k(x-y) - k(x-x_i)| dy \tag{3.15}$$

We estimate $|w(x)|$ for $x \notin \cup_i U_i$ where U_i is the sphere with the same center x_i as B_i but enlarged by a factor $C+1$. In particular if $y \in B_i$ and $x \in U_i^c$, then $|y-x| \geq |x-x_i| - |y-x_i| \geq C|y-x_i|$.

$$\int_{\cap_i U_i^c} |w(x)| dx \leq \sum_i \int_{\cap_i U_i^c} [\int_{B_i} |f(y) - m_j||k(x-y) - k(x-x_i)| dy] dx$$
$$\leq \sum_i \int_{B_i} |f(y) - m_i| [\int_{E_i} |k(x-y) - k(x-x_i)| dx] dy \tag{3.16}$$

where $E_i \subset \{x : |x-y| \geq C|y-x_i|\}$. Therefore,

$$\int_{E_i} |k(x-y) - k(x-x_i)| dx$$
$$\leq \sup_{y,i} \int_{\{x:|x-y| \geq C|y-x_i|\}} |k(x-y) - k(x-x_i)| dx$$
$$\leq \sup_{y} \int_{\{x:|x-y| \geq C|y|\}} |k(x-y) - k(x)| dx$$
$$\leq C \tag{3.17}$$

giving us the estimate

$$\begin{aligned}
\int_{\cap_i U_i^c} |w(x)|dx &\leq C \sum_i \int_{B_i} |f(y) - m_i|dy \\
&\leq C(\|f\|_1 + [\sup_i m_i] \sum_i \mu[B_i]) \\
&\leq C[\|f\|_1 + k_2(d)\ell\mu(G_\ell)] \qquad (3.18) \\
&\leq k_3(d)\|f\|_1] \qquad (3.19)
\end{aligned}$$

7. We can estimate $\mu(\cup_i U_i)) \leq \sum_i \mu(U_i)$ by

$$\sum_i \mu(U_i) \leq k_4(d) \sum_i \mu(B_i) \leq k_5(d)\mu(G_\ell) \leq k_6(d)\frac{\|f\|_1}{\ell}$$

8. We put the pieces together and we are done.

□